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On the development of characteristics in an oscillating, rotating fluid

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Summary

A source of fluid with an oscillatory strength, which is situated on the axis of a rotating fluid, commences to act at time $t = 0$. We describe how inviscid, geostrophic forces lead to the development of the characteristic cone when the frequency of oscillation is less than twice the frequency of rotation. Eventually, viscous forces become important when the time is $O(E^{-1/3})$, where E is the small Ekman number, in forming the thin shear layer along the surface of the cone.

1. Introduction

One of the striking phenomena in rotating fluids is the existence of internal waves due to the oscillation of a body in the fluid. This observation, with clear experimental evidence, was first reported by Oser [1], and is present whenever the frequency of the oscillations is less than twice the frequency of rotation of the fluid. Mathematically, Reynolds [2] showed that in this domain the governing linear equations (based on oscillations with an infinitesimal amplitude) are hyperbolic, and therefore capable of sustaining wave motion, whereas outside the equations are elliptic and the behaviour more similar to potential flow. The experiments clearly show the presence of thin internal shear layers radiating from the body, which are coincident with the characteristic cones expected from the hyperbolic equations, and also with their reflection when such are possible. An introduction to the description of these effects is contained in Greenspan's text [3, p. 200].

The shear layers are represented by the existence of a discontinuity in the solution of the inviscid equations, and it is the role of viscous forces to smooth this discontinuity. The paper by Walton [4], who solved the flow in a split-disc geometry for oscillatory disturbances, showed clearly how viscosity modifies the discontinuities. In fact, one of the clear observations which can be drawn from Walton's calculations concerns the similar structure between these layers and the classical Stewartson layers in a rotating fluid without oscillations.

The aim of the present note is to consider the evolution of these layers through investigating the simple model of a point source of fluid with oscillatory strength, which commences at time $t = 0$ in an unbounded fluid. Although the resulting analytical expressions are fairly complex, the basic mechanism can be clearly delineated. Inviscid, geostrophic forces act through times t less (by an order of magnitude) than $E^{-1/3}$, where

E is the small Ekman number, to focus the distinct effects of the source flow along the characteristic cones, and it is only when $t = O(E^{-1/3})$ that viscous forces become important in the narrow layer, of width $O(E^{1/3})$, which has formed along the cones. The discussion can be seen as a generalization of that included in an earlier paper (Smith [5]), which illustrates further the similarity noted by Walton [4]. There have been few transient solutions for oscillatory flows, and, because they deal with more general situations (c.f. Baines [6]) they become very detailed analytically for just the inviscid models, so viscous effects have, in the past, been inferred rather than calculated.

In the last section we briefly consider the results when the source is placed between parallel discs which rotate with the fluid. It is seen how the development of the reflected layers comes simply through satisfying the requirement that there is no flux across the discs.

2. The fundamental solution

A fluid rotates with constant angular velocity Ω, and the source is placed at the origin O of a cylindrical co-ordinate system where the axis of rotation coincides with the vertical axis. If a is the reference length, we write ar, az as lengths in the radial and axial directions; the time is represented by $\Omega^{-1}t$. We write $\Omega au(r, z, t)$, $\Omega av(r, z, t)$ and $\Omega aw(r, z, t)$ for the radial, azimuthal and axial velocities respectively; the pressure is $\rho\Omega^2a^2p(r, z, t)$ when the constant density of the fluid is ρ. If the strength of the source is ϵ, which is small enough to permit linearization of the equations, then writing $u = \epsilon U$, $v = r + \epsilon V$, $w = \epsilon W$, $p = \frac{1}{2}r^2 + \epsilon P$, we have

$$U_r + \frac{1}{r}U + W_z = \frac{1}{r}\delta(r)\delta(z)H(t)\cos\omega t, \tag{2.1}$$

$$U_t - 2V = -P_r + E\left(U_{rr} + \frac{1}{r}U_r - \frac{1}{r^2}U + U_{zz}\right), \tag{2.2}$$

$$V_t + 2U = E\left(V_{rr} + \frac{1}{r}V_r - \frac{1}{r^2}V + V_{zz}\right), \tag{2.3}$$

$$W_t = -P_z + E\left(W_{rr} + \frac{1}{r}W_r + W_{zz}\right); \tag{2.4}$$

E is the Ekman number, defined by $E = \nu/\Omega a^2$, where ν represents the kinematic viscosity. The only conditions to be imposed are that the velocities U, V, $W \to 0$ as r, $z \to \infty$.

The solution of these equations is developed in the same manner as that done previously in Smith [5], through taking Laplace transforms in t, Fourier transforms in z and Hankel transforms in r. For example, when

$$\bar{V}(k, \alpha, s) = k\int_0^\infty e^{-st}dt\int_0^\infty \cos\alpha z dz\int_0^\infty rV(r, z, t)J_1(kr)dr, \tag{2.5}$$

then the calculations show

$$\bar{V} = -\frac{2k^2 s\{E(k^2+\alpha^2)+s\}}{(s^2+\omega^2)\left[\{E(k^2+\alpha^2)+s\}^2(k^2+\alpha^2)+4\alpha^2\right]}; \tag{2.6}$$

similar expressions follow for U, W and P. These will be valid for all values of E, though we now proceed to consider asymptotic approximations for $E \ll 1$. The other assumption we set is that the frequency ω satisfies $\omega < 2$, the situation with $\omega > 2$ is of less interest and is not pursued here.

It is most convenient, when calculating the approximations, to work with the function $X(r, z, t)$ defined by $X_r = V$-generalized functions, which would otherwise occur, are now avoided without losing the essential phenomena we intend to describe. Hence, much of the remainder of this paper discusses asymptotic results for the integral

$$X = \frac{2}{\pi^2 \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{s\,\mathrm{e}^{st}\mathrm{d}s}{s^2+\omega^2} \int_0^\infty \cos\alpha z\,\mathrm{d}\alpha \int_0^\infty kJ_0(kr)$$
$$\times \frac{\{E(k^2+\alpha^2)+s\}\mathrm{d}k}{\{E(k^2+\alpha^2)+s\}^2(k^2+\alpha^2)+4\alpha^2}. \tag{2.7}$$

Firstly, we set $E=0$ to gain the inviscid solution, which is written as $\chi(r, z, t) = \lim_{E\to 0} X(r, z, t)$. When the inverse Fourier, and then Hankel, transforms are taken, it follows that

$$\chi = \frac{2}{\pi \mathrm{i} R} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{s\,\mathrm{e}^{st}\mathrm{d}s}{(s^2+\omega^2)(s^2+\sigma^2)^{1/2}}, \tag{2.8}$$

where $R^2 = r^2+z^2$, and σ is the basic geometric similarity variables $\sigma = 2r/R$. The integral (2.8) can be evaluated as the double series

$$\chi = \frac{2}{R}\sum_{n=0}^{\infty} \frac{(-1)^{n+1} t^{2n+1}}{(2n+1)!} \sum_{m=0}^{n} \frac{(2m)!}{2^{2m}(m!)^2}\omega^{2(n-m)}\sigma^{2m},$$

though this is of value only for small t. However, through deforming the contour of the integral (2.8), taking care to adjust for the branch points at $s = \pm \mathrm{i}\sigma$, or, alternatively, through taking the inverse Laplace, and then Fourier transform of the original triple integral, plus a change of variable of integration, we obtain

$$\chi = \frac{4}{\pi R}\int_0^\sigma \frac{\omega \sin\omega t - \beta \sin\beta t}{(\beta^2-\omega^2)(\sigma^2-\beta^2)^{1/2}}\mathrm{d}\beta; \tag{2.9}$$

it is this integral which reveals most clearly the behaviour for large t.

Now the dominant contribution to (2.9) as $t \to \infty$, when $|\omega - \sigma| = O(1)$, comes from just the first term in the numerator, which shows

$$\chi = \begin{cases} -\dfrac{2\sin\omega t}{R(\omega^2-\sigma^2)^{1/2}}, & \omega > \sigma, \\ 0, & \omega < \sigma. \end{cases} \tag{2.10}$$

The characteristic is along $\sigma = \omega$–that is, where $r/z = \omega(4-\omega^2)^{-1/2}$, as shown by Reynolds [2]; the velocities are zero outside the characteristic cone, but become infinite on the surface.

Next, through adding and subtracting the term $\omega \sin \beta t$ to the numerator of (2.9), and simplifying, we see that

$$\chi = -\frac{4}{\pi R}\left[\int_0^{\sigma} \frac{\sin\beta t\cdot \mathrm{d}\beta}{(\beta+\omega)(\sigma^2-\beta^2)^{1/2}} + \omega\sin\omega t\int_{\omega-\sigma}^{\omega}\frac{(1-\cos\beta t)\mathrm{d}\beta}{\beta(2\omega-\beta)\left\{\sigma^2-(\omega-\beta)^2\right\}^{1/2}}\right.$$

$$\left. + \omega\cos\omega t\int_{\omega-\sigma}^{\omega}\frac{\sin\beta t\cdot\mathrm{d}\beta}{\beta(2\omega-\beta)\left\{\sigma^2-(\omega-\beta)^2\right\}^{1/2}}\right]$$

exactly. For large t, the first integral is $O(t^{-1})$, but when

$$\xi = (\omega-\sigma)t = O(1)$$

the contribution from the second and third integrals leads to

$$\chi \simeq -\frac{1}{\pi R}\left(\frac{2t}{\omega}\right)^{1/2}\left[\sin\omega t\int_{\xi}^{\infty}\frac{(1-\cos\gamma)\mathrm{d}\gamma}{\gamma(\gamma-\xi)^{1/2}} + \cos\omega t\int_{\xi}^{\infty}\frac{\sin\gamma\cdot\mathrm{d}\gamma}{\gamma(\gamma-\xi)^{1/2}}\right]. \tag{2.11}$$

Hence, the quantity ξ is the basic similarity variable which describes the formation of the characteristic cone in the fluid, across which there is the discontinuity in the inviscid description as $t \to \infty$.

The integrals in (2.11) have different values depending on whether ξ is positive or negative; for $\xi > 0$ it can be shown that (2.11) is given exactly by

$$\chi \simeq -\frac{1}{R}\left(\frac{2t}{\omega\xi}\right)^{1/2}\left[\{C(\xi)+S(\xi)\}\sin\omega t + \{C(\xi)-S(\xi)\}\cos\omega t\right], \tag{2.12}$$

where $C(x)$ and $S(x)$ are Fresnel integrals defined by

$$C(x) + \mathrm{i}S(x) = (2\pi)^{-1/2}\int_0^x \gamma^{-1/2}\,\mathrm{e}^{\mathrm{i}\gamma}\mathrm{d}\gamma.$$

Further, for $\zeta = -\xi > 0$, it also follows that

$$\chi \simeq \frac{1}{R}\left(\frac{2t}{\omega\zeta}\right)^{1/2}\Bigg[\Bigg[\{1 - C(\zeta) - S(\zeta)\} \cos 2\zeta + \{C(\zeta) - S(\zeta)\} \sin 2\zeta$$

$$- \left(\frac{2\zeta}{\pi}\right)^{1/2}(\cos\zeta + \sin\zeta) S_{0,1/2}(\zeta)\Bigg] \sin \omega t - \Bigg[1 + \{C(\zeta) - S(\zeta)\} \cos 2\zeta$$

$$- \{1 - C(\zeta) - S(\zeta)\} \sin 2\zeta + \left(\frac{2\zeta}{\pi}\right)^{1/2}(\cos\zeta - \sin\zeta) S_{0,1/2}(\zeta)\Bigg] \cos \omega t\Bigg],$$

where $S_{0,1/2}(x)$ is a Lommel function defined by

$$S_{0,1/2}(x) = 2\int_0^\infty \cosh\gamma \, e^{-x \sinh 2\gamma} d\gamma;$$

c.f. Erdelyi et al. [7].

Some special cases can be noted. As $\xi \to \infty$, then $C(\xi)$, $S(\xi) \to 1/2$, so that

$$\chi \simeq -\frac{1}{R}\left(\frac{2t}{\omega\xi}\right)^{1/2} \sin \omega t,$$

to agree with (2.10). When $\zeta \to \infty$, it follows from the property $S_{0,1/2}(x) \simeq \frac{1}{2}(\pi + 2)x^{-1}$ as $x \to \infty$, that χ tends to zero as necessary, with a decay proportional to ζ^{-1}. Also, when the limits as both ξ and ζ tend to zero are taken, it is seen after a more detailed calculation, using, in particular, the property $S_{0,1/2}(x) \simeq \frac{1}{2}(2\pi)^{1/2}x^{-1/2}$ as $x \to 0+$, that χ is continuous with

$$\chi = -\frac{2}{R}\left(\frac{2t}{\pi\omega}\right)^{1/2} \sin\left(\omega t + \tfrac{1}{4}\pi\right) \qquad \text{for} \quad \sigma = \omega. \tag{2.14}$$

Consequently, the magnitude of χ tends to infinity as $O(t^{1/2})$ on the characteristic cone; also, the oscillation has a phase shift from that present within the cone by (2.10).

This inviscid solution can be expressed by $X = R^{-1}f(\sigma, t)$ for all time, so the physical variables are $U = R^{-2}\mathscr{U}(\sigma, t)$, $V = R^{-2}\mathscr{V}(\sigma, t)$, $W = zR^{-3}\mathscr{W}(\sigma, t)$, with the stream function $\Psi = zR^{-1}\psi(\sigma, t)$. It then follows that $\psi = -\frac{1}{2}f_t$, so that $\mathscr{V} = \frac{1}{2}\{(4 - \sigma^2)f_\sigma - \sigma f\}$, plus $\mathscr{U} = -\frac{1}{2}\mathscr{V}_t$ and $\mathscr{W} = -\sigma^{-1}\mathscr{V}_t$; the resulting equation away from the origin for ψ, in terms of the similarity variable σ, is

$$\sigma(4 - \sigma^2)\psi_{\sigma\sigma tt} - 2(2 + \sigma^2)\psi_{\sigma tt} + \sigma^3(4 - \sigma^2)\psi_{\sigma\sigma} + \sigma^2(8 - 5\sigma^2)\psi_\sigma - 3\sigma^3\psi = 0.$$

Finally, the relation between $\mathscr{U}$ and $\mathscr{W}$ is equivalent to $U/W = r/z$, so that the inviscid theory shows the velocity in the azimuthal plane to be completely radial (in the spherical sense), as could have been anticipated. It is the role of the geostrophic forces to weaken the magnitude of this velocity outside the characteristic cone as the time increases.

It is more difficult to evaluate the integrals when the viscous effects are included. However, when we take the large-time solution, after the transient effects have decayed, then it is seen that

$$\chi \simeq \frac{2 \sin \omega t}{(4-\omega^2)^{1/2}} \int_0^\infty \sin \frac{k\omega z}{(4-\omega^2)^{1/2}} \exp\left\{-\frac{16Ek^3 z}{(4-\omega^2)^{5/2}}\right\} J_0(kr)\mathrm{d}k, \tag{2.15}$$

after calculating the residues for the α-integral when $E \ll 1$– which agrees with (2.10) when $E = 0$. In the layer along the characteristic defined by $\eta = O(1)$, where

$$(4-\omega^2)^{1/2} - \omega z = E^{1/3}\eta,$$

the integral (2.15) becomes

$$\left(\frac{2}{\pi r}\right)^{1/2} E^{-1/6} |\eta|^{-1/2} \sin \omega t \int_0^\infty \gamma^{-1/2} \sin\left(\gamma + \frac{1}{4}\pi\right) \exp\left\{-\frac{16\gamma^3 z}{(4-\omega^2)^2 \eta^3}\right\} \mathrm{d}\gamma;$$

hence the viscous layer has width $O(E^{1/3})$, as shown by Walton [4], with $X = O(E^{-1/6})$ there.

Further, we can evaluate the inverse Laplace transform in (2.7) exactly, then set $E \ll 1$ following transformations similar to those pursued earlier in the inviscid case, to show

$$X \simeq \frac{4}{\pi} \int_0^2 \frac{\mathrm{d}\beta}{\beta(\omega^2 - \beta^2)} \int_0^\infty \cos \alpha z J_0\left\{\beta^{-1}(4-\beta^2)^{1/2} \alpha r\right\}$$
$$\times \left[\omega \sin \omega t - \beta\, \mathrm{e}^{-4E\alpha^2\beta^{-2}} \sin \beta t\right] \mathrm{d}\alpha;$$

this reduces to (2.9) quickly when $E = 0$. For small E the dominant part of the α-integral is derived where α is large, and so can be evaluated on approximating the Bessel function (though only in terms of confluent hypergeometric functions, which does not help particularly). Nevertheless, the contribution for $\tau = E^{1/3}t = O(1)$ comes from the neighbourhood of $\beta = \omega$, which reveals the basic parameter to be $\eta^2\tau^{-1}$ and X to be $O(E^{-1/6})$ when η, $\tau = O(1)$.

Consequently, the characteristic cone is formed through inviscid action only during the time $t \ll E^{-1/3}$, with the flow characterized by the similarity variable ξ. When $t = O(E^{-1/3})$ viscous forces come to act within the layer where $\eta = O(1)$ to develop the final structure in the layer on the surface of the cone; when $t \gg E^{-1/3}$, the simple oscillatory state alone dominates.

3. Source between parallel discs

To extend the fundamental solution we now briefly consider the effects when the source at the origin is placed between infinite discs which, for convenience, are symmetrically placed along $z = \pm d$; the discs rotate with angular velocity Ω. Following the earlier approach (Smith [5]) we write, for example,

$$W = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{s\,\mathrm{e}^{st}\mathrm{d}s}{s^2+\omega^2} \int_0^{\infty} \{ k\,\mathrm{e}^{-\theta z} + C(k, s) \sinh \theta z \} J_0(kr)\mathrm{d}k, \tag{3.1}$$

where $\theta = ks(s^2+4)^{-1/2} + 16Eks(s^2+4)^{-5/2} + O(E^2)$, and $C(k, s)$ is an unknown function to be calculated on satisfying the boundary conditions on $z = \pm d$; equivalent expressions follow for U, V, P. They are valid everywhere in the flow region except in the Ekman layers along the discs; however, the no-slip conditions there can be replaced by the Ekman compatibility condition $W = -\frac{1}{4}E^{1/2}(P_{rr} + r^{-1}P_r)$. When the details are completed, it is seen that the dominant behaviour away from the characteristic surfaces is completely inviscid for small E, that the Ekman condition reduces to just $W = 0$ on $z = \pm d$, and hence that it is now sufficient to take $\theta \simeq ks(s^2+4)^{-1/2}$. Consequently, $C(k, s) \simeq -k\,\mathrm{e}^{-\theta d}/\sinh \theta d$, and the corresponding expression for the function $\chi(r, z, t)$ is

$$\chi \simeq \frac{1}{\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{s\,\mathrm{e}^{st}\mathrm{d}s}{(s^2+\omega^2)(s^2+4)^{1/2}} \int_0^{\infty} \left[\mathrm{e}^{-\theta z} + \frac{\mathrm{e}^{-\theta d}\cosh \theta z}{\sinh \theta d} \right] J_0(kr)\mathrm{d}k. \tag{3.2}$$

It is now a straightforward calculation to expand the expression in the square brackets of (3.2) as an infinite series of exponentials, and evaluate the k-integral, to show

$$\chi \simeq \frac{1}{\pi \mathrm{i}} \sum_{n=-\infty}^{\infty} \frac{1}{R_n} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{s\,\mathrm{e}^{st}}{s^2+\omega^2} \left(s^2+\sigma_n^2\right)^{-1/2} \mathrm{d}s, \tag{3.3}$$

where $R_n^2 = r^2 + (z+2nd)^2$ and $\sigma_n = 2r/R_n$. The term corresponding to $n = 0$ gives the original source, and the other terms represent an infinite series of reflections in the planes $z = \pm d$. The final expression for χ is found on evaluating (3.3), which is just a sum of the integrals (2.8).

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